IMM 403
June 1974

Courant Institute of
Mathematical Sciences

Construction of a Three Dimensional Displacement Field from a Solution to the von Karman Equations

Chester B. Sensenig

Prepared under Contract N00014-67-A-0467-0030
with the Office of Naval Research

Distribution of this document is unlimited.

New York University

NEW YORK UNIVERSITY
COURANT INSTITUTE-LIBRARY
251 Mercer St. New York, N.Y. 10012

c.1

IMM 403
June 1974

New York University

Courant Institute of Mathematical Sciences

CONSTRUCTION OF A THREE DIMENSIONAL DISPLACEMENT FIELD FROM A SOLUTION TO THE VON KARMAN EQUATIONS

Chester B. Sensenig

Results obtained at the Courant Institute of Mathematical Sciences, New York University, with the Office of Naval Research, Contract N00014-67-A-0467-0030.

Reproduction in whole or in part is permitted for any purpose of the United States Government.

Distribution of this document is unlimited.

Introduction

In this report we define a three dimensional displacement field in terms of a solution to the von Karman equations and determine how well the three dimensional displacement field satisfies the exact theory of elasticity for a plate with zero body force vector and zero stress vector at the faces of the plate. We do this by getting estimates for the derivatives of a solution of the von Karman equations (1.3) (where w is the nondimensional vertical displacement and ψ is a nondimensional stress function), and then we use these estimates and the defined displacement field in the exact theory for an isotropic hyperelastic material to compute the order of the body force vector and stress vector on the faces corresponding to the given displacement field. Letting F_1, F_2, F_3 be the components of the body force vector (force per unit undeformed volume) with respect to the $\bar{x}_1, \bar{x}_2, \bar{x}_3$ axes (see Fig. 1), and letting T_1, T_2, T_3 be the components of the stress vector on the faces (force per unit undeformed area), we obtain

$$(0.1)\qquad \left\{\begin{aligned} F_1, F_2 &= O\left(\frac{\mu h^4}{D^5}\,\varepsilon^2\right)\\ F_3 &= O\left(\frac{\mu h^5}{D^6}\,\varepsilon^2\right)\\ T_1, T_2 &= O\left(\mu\,\frac{h^5}{D^5}\,\varepsilon^2\right)\\ T_3 &= O\left(\frac{\mu h^6}{D^6}\,\varepsilon^2\right)\end{aligned}\right.$$

where μ is one of the Lamé constants, 2h is the thickness of the undeformed plate, D is the diameter of the undeformed plate, and ε measures the size of the solution w, ψ to the

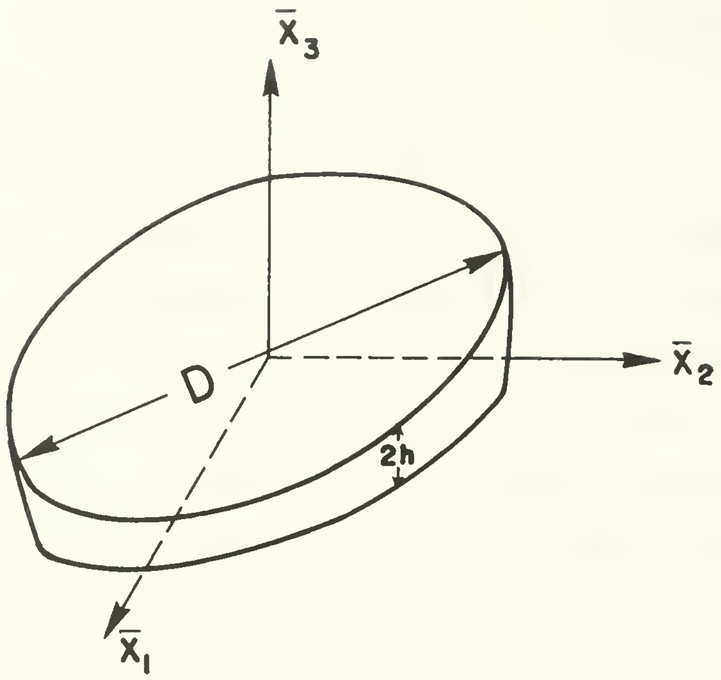

fig. 1

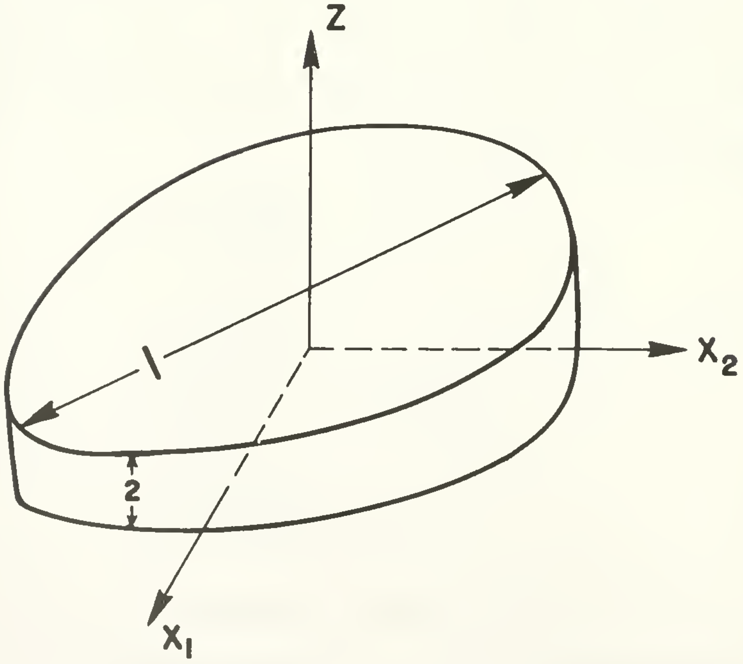

fig. 2

von Karman equations and is defined in two different ways in Section 2. It should be observed that ε may be taken to be independent of h/D since our nondimensional von Karman equations (1.3) are independent of h/D. Consequently, we can let ε and h/D vary independently in (0.1) as well as in the following results.

In a previous report [2], the displacements for the exact theory were expressed in terms of the vertical deflection of the middle surface and a stress function. It was expected that those formulas for the displacements could be used here to define a displacement field. However, changes were made in the formulas used here since the stress function is now an exact solution to the von Karman equations rather than an approximate solution. The method used to obtain the formulas for the displacements is the same, however, as used in [2]. We simply exhibit the formulas in (1.6) rather than repeat the derivation.

The criterion used in deciding how many terms to include in the formulas for the displacements was that the body force vector and stress vector at the faces should have as small an order as possible without needing plate equations with more terms than the von Karman equations. Although the von Karman equations involve only two elastic constants, it is interesting that this criterion leads us to include terms in the formulas for the displacements which include five elastic constants. In particular if we dropped all terms in the vertical displacement of the order of the terms containing the additional three elastic constants, then the vertical components of the stress vector at the faces would be of order $\mu(h^4/D^4)\varepsilon^2$ rather than $\mu(h^6/D^6)\varepsilon^2$.

1. The von Karman Equations and Displacement Field.

Let x_1, x_2, z be nondimensional variables given by

$$x_\alpha = \frac{\bar{x}_\alpha}{D}, \quad z = \frac{\bar{x}_3}{h}. \tag{1.1}$$

Then the domain of (x_1, x_2) has diameter "one" with the origin in its interior and $|z| \leq 1$ (see Fig. 2).

Let ψ and w be nondimensional functions of x_1 and x_2 (the stress function and vertical displacement of the middle surface), and let $\tau_{\alpha\beta}$ (the horizontal stresses on the middle surface) be given by

$$\tau_{11} = \frac{\partial^2 \psi}{\partial x_2^2}; \quad \tau_{22} = \frac{\partial^2 \psi}{\partial x_1^2}; \quad \tau_{12} = \tau_{21} = -\frac{\partial^2 \psi}{\partial x_1 \, \partial x_2}. \tag{1.2}$$

As usual let Greek subscripts take on the values 1, 2 and Latin subscripts the values 1, 2, 3.

Let w and ψ satisfy the well known von Karman equations

$$\left\{ \begin{aligned} \Delta^2 \psi &= \frac{3\lambda+2\mu}{4(\lambda+\mu)} [w_{,\gamma\nu} w_{,\gamma\nu} - (\Delta w)^2] \\ \Delta^2 w &= \frac{3(\lambda+2\mu)}{2(\lambda+\mu)} w_{,\gamma\nu} \tau_{\gamma\nu} \end{aligned} \right. \tag{1.3}$$

where λ and μ are the Lamé elastic constants,

$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian operator,

$\Delta^2 = \frac{\partial^4}{\partial x_1^4} + 2 \frac{\partial^4}{\partial x_1^2 \, \partial x_2^2} + \frac{\partial^4}{\partial x_2^4}$ is the biharmonic operator,

$w_{,\gamma\nu} = \frac{\partial^2 w}{\partial x_\gamma \, \partial x_\nu}$ etc., and repeated subscripts are summed over their range.

We define nondimensional horizontal displacements of the middle surface, v_γ, as the unique solutions of the equations

$$(1.4)\quad v_{\gamma,\alpha\beta} = -w_{,\gamma}w_{,\alpha\beta} + \tau_{\beta\gamma,\alpha} + \tau_{\alpha\gamma,\beta} - \tau_{\alpha\beta,\gamma} - \frac{\lambda}{3\lambda+2\mu}(\Delta\psi_{,\alpha}\delta_{\beta\gamma} + \Delta\psi_{,\beta}\delta_{\alpha\gamma} - \Delta\psi_{,\gamma}\delta_{\alpha\beta})$$

such that at the origin

$$(1.5)\quad \begin{cases} v_\gamma = 0 \\ v_{\alpha,\beta} = -\frac{1}{2}w_{,\alpha}w_{,\beta} + \tau_{\alpha\beta} - \frac{\lambda}{3\lambda+2\mu}\delta_{\alpha\beta}\Delta\psi . \end{cases}$$

That unique functions exist satisfying (1.4) and (1.5) follows from standard compatability theorems such as the one in [3]. Letting

$$\psi_{\gamma\alpha\beta} = -w_{,\gamma}w_{,\alpha\beta} + \tau_{\beta\gamma,\alpha} + \tau_{\alpha\gamma,\beta} - \tau_{\alpha\beta,\gamma} - \frac{\lambda}{3\lambda+2\mu}(\Delta\psi_{,\alpha}\delta_{\beta\gamma} + \Delta\psi_{,\beta}\delta_{\alpha\gamma} - \Delta\psi_{,\gamma}\delta_{\alpha\beta}) ,$$

the compatability equations for (1.4) are $\psi_{\gamma\alpha\beta} = \psi_{\gamma\beta\alpha}$ and $\psi_{\gamma\alpha\beta,\nu} = \psi_{\gamma\alpha\nu,\beta}$. Each of these equations is either trivially true or reduces to the first equation of (1.3) so that the compatability conditions are satisfied.

We define a three dimensional displacement field by

$$
\left\{
\begin{aligned}
\mu_\alpha &= \frac{h^2}{D}(v_\alpha - zw_{,\alpha}) + \frac{h^4}{D^3}\Big\{\frac{\lambda}{2(3\lambda+2\mu)} z^2 \Delta\psi_{,\alpha} \\
&\quad + \Big[\frac{3\lambda+4\mu}{6(\lambda+2\mu)} z^3 - \frac{2(\lambda+\mu)}{\lambda+2\mu} z\Big]\Delta w_{,\alpha} + z v_{\gamma,\alpha} w_{,\gamma} + \frac{\lambda}{3\lambda+2\mu} z\Delta\psi w_{,\alpha} \\
&\quad - \frac{\lambda}{2(\lambda+2\mu)} z^2 \Delta w\, w_{,\alpha} + \frac{z}{2} w_{,\gamma} w_{,\gamma} w_{,\alpha}\Big\} \\
\mu_3 &= hw + \frac{h^3}{D^2}\Big[-\frac{\lambda}{3\lambda+2\mu} z\Delta\psi + \frac{\lambda}{2(\lambda+2\mu)} z^2 \Delta w - \frac{z}{2} w_{,\gamma} w_{,\gamma}\Big] \\
&\quad + \frac{h^5}{D^4}\Big\{-\frac{\lambda}{2(\lambda+2\mu)} z v_{\gamma,\nu} v_{\gamma,\nu} + C_1 z(\Delta\psi)^2 - \frac{B}{\lambda+2\mu} z\psi_{,\gamma\nu}\psi_{,\gamma\nu} \\
&\quad + C_2 z^2 \Delta\psi\ \Delta w + \frac{2\lambda(\lambda+\mu)}{(\lambda+2\mu)(3\lambda+2\mu)} z^2 w_{,\gamma}\Delta\psi_{,\gamma} + \Big[\frac{3\lambda+\ \mu+2B}{2(\lambda+2\mu)} z^2 \\
&\quad - \frac{3\lambda+2\mu}{16(\lambda+\mu)} z^4\Big] w_{,\gamma\nu}\tau_{\gamma\nu} - \frac{\lambda}{2(3\lambda+2\mu)} z\Delta\psi w_{,\gamma} w_{,\gamma} + z\tau_{\gamma\nu} w_{,\gamma} w_{,\nu} \\
&\quad + \Big(C_3 z^3 - \frac{\mu}{\lambda+2\mu} z\Big) w_{,\gamma\nu} w_{,\gamma\nu} + \Big[C_4 z^3 - \frac{\lambda\mu}{(\lambda+2\mu)^2} z\Big](\Delta w)^2 \\
&\quad + \Big[\frac{3\lambda+4\mu}{6(\lambda+2\mu)} z^3 - \frac{2(\lambda+\mu)}{\lambda+2\mu} z\Big] w_{,\gamma}\Delta w_{,\gamma} - \frac{z}{8} w_{,\gamma} w_{,\gamma} w_{,\nu} w_{,\nu}\Big\}
\end{aligned}
\right.
\tag{1.6}
$$

where

$$
\left\{
\begin{aligned}
C_1 &= -\frac{\lambda^2}{2(3\lambda+2\mu)^2} - \frac{4\mu^2 A - \lambda(3\lambda+8\mu)B + \lambda^2 C}{(\lambda+2\mu)(3\lambda+2\mu)^2} \\
C_2 &= \frac{4\mu^2 A - 6\lambda\mu B + \lambda^2 C}{(\lambda+2\mu)^2(3\lambda+2\mu)} \\
C_3 &= \frac{-5\lambda^2 + 4\lambda\mu + 8\mu^2}{24(\lambda+\mu)(\lambda+2\mu)} - \frac{B}{3(\lambda+2\mu)} \\
C_4 &= \frac{\lambda(\lambda^2 + 10\lambda\mu + 8\mu^2)}{24(\lambda+\mu)(\lambda+2\mu)^2} - \frac{4\mu^2 A + \lambda(\lambda-4\mu)B + \lambda^2 C}{3(\lambda+2\mu)^3}
\end{aligned}
\right.
\tag{1.7}
$$

The parameters A, B, C appearing in (1.6) and (1.7) are elastic constants whose connections with the elastic material are given below in (1.9).

The notation $,\alpha$ denotes $\partial/\partial x_\alpha$ when applied to the nondimensional quantities w, ψ, v_α, $\tau_{\alpha\beta}$. However, $,\alpha$ or $,i$

denotes $\partial/\partial\bar{x}_\alpha$ or $\partial/\partial\bar{x}_i$ when applied to the dimensional displacements u_j defined in (1.6), the strains e_{jk} defined in (1.8), or the stress quantities t_{jk}, q_{jk} defined in (1.10).

The strains are defined by

$$(1.8) \qquad e_{ij} = \frac{1}{2}\left(u_{i,j}+u_{j,i}+u_{k,i}\,u_{k,j}\right) .$$

The nonlinear dependence of the strains on the displacement gradient gives rise to the nonlinear terms in the von Karman equations.

For an isotropic hyperbolic material the strain energy density ,W (strain energy per unit undeformed volume), has the form

$$(1.9) \quad W = \frac{\lambda}{2} e_{ii}e_{jj} + \mu e_{ij}e_{ij} + \frac{A}{3} e_{ii}e_{jj}e_{kk} + B\, e_{ii}e_{jk}e_{jk} + \frac{C}{3} e_{ij}e_{jk}e_{ki} + \text{4th order terms in the strains.}$$

Our previously mentioned criterion leads us to go up to cubic terms in the expansion of W, and we are assuming that W has as many continuous derivatives with respect to the strains as are needed to justify (1.9) and also the remainder of the work.

If we let t_{ij} be the components of the second Piola-Kirchhoff tensor and q_{ij} be the components of the first Piola-Kirchoff tensor (see [4]), then

$$(1.10)\quad \begin{cases} t_{ij} = \dfrac{\partial w}{\partial e_{ij}} = (\lambda e_{kk} + Ae_{kk}e_{\ell\ell} + Be_{k\ell}e_{k\ell} + A_3)\delta_{ij} \\ \qquad + 2(\mu + Be_{kk} + A_2)e_{ij} + (C+A_1)e_{ik}e_{kj} \\ q_{ij} = \dfrac{\partial w}{\partial \mu_{i,j}} = t_{ij} + \mu_{i,k}t_{kj} \end{cases}$$

where A_1, A_2, A_3 denote expressions of order 1,2,3 in the strains.

The body force vector (F_1, F_2, F_3) (force per unit undeformed volume) and the stress vector at a face (T_1, T_2, T_3) (force per unit undeformed area) are given by the equilibrium equations and boundary conditions at the faces (see [4]).

$$(1.11)\quad \begin{cases} F_i = -q_{ij,j} \\ T_i = q_{i3} \quad \text{for } z = 1 \text{ or } \bar{x}_3 = h \\ T_i = -q_{i3} \quad \text{for } z = -1 \text{ or } \bar{x}_3 = -h \end{cases}$$

2. Estimates for the Derivatives of a Solution to the von Karman Equations.

Let $(\hat{x}_1, \hat{x}_2)$ be any fixed interior point of our nondimensional horizontal domain and let $r = \sqrt{(x_1-\hat{x}_1)^2 + (x_2-\hat{x}_2)^2}$. For convenience we introduce

$$(2.1)\qquad \zeta = \begin{cases} \left(1 - \dfrac{r^2}{\theta^2}\right)^2 & \text{for} \quad r \leq \theta \\ 0 & \text{for} \quad r \geq \theta \end{cases}$$

where θ is the distance (nondimensional) from $(\hat{x}_1, \hat{x}_2)$ to the boundary of the horizontal domain.

If S is a set of functions defined on our nondimensional horizontal domain, we let $\dot{S}$ be the set of functions which are first derivatives of elements of S, $\ddot{S}$ be the set of functions which are second derivatives of elements of S, etc.

If T is another set of functions defined on our horizontal nondimensional domain, we let ST be the set of all functions which are a product of an element of S and an element of T.

If S is a finite set of functions defined on our horizontal nondimensional domain, we let $|S|$ be the square root of the sum of the squares of all elements of S, and we define $\|S\| \geq 0$ as

$$\|S\|^2 = \int\int |S|^2 \, dx_1 \, dx_2$$

where the integration is taken over our nondimensional horizontal domain.

We ambiguously let ζ denote both the function defined in (2.1) and the set whose one element is the function defined in (2.1). Then $\dot{\zeta}$, $\ddot{\zeta}$, etc. have meaning as the set of all first derivatives of ζ, the set of all second derivatives of ζ, etc.

With the above notation we introduce the order relations

$$(2.2) \qquad \left\{ \begin{array}{l} |\zeta| = O(1) \ , \quad |\dot{\zeta}| = O(1) \ , \quad |\ddot{\zeta}| = O(1) \\ \|\dot{\zeta} \; \dot{S}\| = O(\|S\|) + o(\|\zeta \; \Delta S\|) \\ \|\zeta \; \dot{S}\| = O(\|S\|) + o(\|\zeta \; \Delta S\|) \\ \|\zeta \; \ddddot{S}\| = O(\|\ddot{S}\| + o \; \|\zeta \; \Delta^2 S\|) \\ |S|_0 = O(\|S\| + \|\zeta \; \ddot{S}\|) \ . \end{array} \right.$$

The first line of (2.2) follows from (2.1).

The notation $\|S\| = O(\|T\|)$ means there is a constant k, which may depend on θ but not on the elements of S and T, such that $\|S\| \leq k\|T\|$.

The notation $\|S\| = O(\|T\|) + o\|U\|$ means that for each k there is a K, which may depend on θ but not on the elements of S, T, U, such that $\|S\| \leq K\|T\| + k\|U\|$.

$\Delta^2 S$ is the set of functions which are the result of applying Δ^2 to each element of S

$|S|_0$ denotes the value of $|S|$ at $r = 0$.

The estimates (2.2) can be proved by techniques similar to those used in the Appendix to [1], and are valid if the elements of S have five continuous derivatives (needed for the fourth line). We will prove only the fourth line here.

Let s_k $(k = 1,2,\ldots,n)$ be the elements of S. Using the divergence theorem and the fact that $\zeta = 0$ on the boundary, we have

$$\|\zeta \ddddot{S}\|^2 = \int\int \zeta^2 \, s_{k,\alpha\beta\gamma\nu} \, s_{k,\alpha\beta\gamma\nu} \, dx_1 \, dx_2$$

$$= \iint [(\zeta^2 s_{k,\alpha\beta\gamma\nu} s_{k,\alpha\beta\gamma})_{,\nu} - \zeta^2 \Delta s_{k,\alpha\beta\gamma} s_{k,\alpha\beta\gamma}$$

$$- (\zeta^2)_{,\nu} s_{k,\alpha\beta\gamma\nu} s_{k,\alpha\beta\gamma}] \, dx_1 \, dx_2$$

$$= - \iint [\zeta^2 \Delta s_{k,\alpha\beta\gamma} s_{k,\alpha\beta\gamma} + (\zeta^2)_{,\nu} s_{k,\alpha\beta\gamma\nu} s_{k,\alpha\beta\gamma}] \, dx_1 \, dx_2$$

$$= \iint [-(\zeta^2 \Delta s_{k,\alpha\beta} s_{k,\alpha\beta\gamma})_{,\gamma} + \zeta^2 \Delta s_{k,\alpha\beta} \Delta s_{k,\alpha\beta} +$$

$$+ (\zeta^2)_{,\gamma} \Delta s_{k,\alpha\beta} s_{k,\alpha\beta\gamma} + \text{etc.}] \, dx_1 \, dx_2$$

$$= \iint [(\zeta^2 \Delta s_{k,\alpha\beta} \Delta s_{k,\alpha})_{,\beta} - \zeta^2 \Delta^2 s_{k,\alpha} \Delta s_{k,\alpha}$$

$$- (\zeta^2)_{,\beta} \Delta s_{k,\alpha\beta} \Delta s_{k,\alpha} + \text{etc.}]\, dx_1\, dx_2$$

$$= \iint [-(\zeta^2 \Delta^2 s_k \Delta s_{k,\alpha})_{,\alpha} + \zeta^2 \Delta^2 s_k \Delta^2 s_k +$$

$$+ (\zeta^2)_{,\alpha} \Delta^2 s_k \Delta s_{k,\alpha} + \text{etc.}]\, dx_1\, dx_2$$

$$= \iint [\zeta^2 \Delta^2 s_k \Delta^2 s_k + (\zeta^2)_{,\alpha} \Delta^2 s_k \Delta s_{k,\alpha} - (\zeta^2)_{,\beta} \Delta s_{k,\alpha\beta} \Delta s_{k,\alpha}$$

$$+ (\zeta^2)_{,\gamma} \Delta s_{k,\alpha\beta} s_{k,\alpha\beta\gamma} - (\zeta^2)_{,\nu} s_{k,\alpha\beta\gamma\nu} s_{k,\alpha\beta\gamma}]\, dx_1\, dx_2$$

$$= O(\| \zeta\ \Delta^2 S \|^2 + \| \dot{\zeta} \dddot{S} \|^2) + o(\| \zeta \ddddot{S} \|^2)$$

making repeated use of $|a\ b| = O(|a|^2) + o(|b|^2)$ and the first line of (2.2). Hence

$$\| \zeta \ddddot{S} \| = (\| \zeta\ \Delta^2 S \| + \| \dot{\zeta}\ \dddot{S} \|) + o(\| \zeta \ddddot{S} \|) .$$

From line 2 of (2.2)

$$\| \dot{\zeta}\ \dddot{S} \| = O(\| \ddot{S} \|) + o(\| \zeta \ddddot{S} \|)$$

so that

$$\| \zeta\ \ddddot{S} \| = O(\| \zeta\ \Delta^2 S \| + \| \ddot{S} \|) + o \| \zeta \ddddot{S} \|$$

or

$$\| \zeta\ \ddddot{S} \| = O(\| \ddot{S} \| + \| \zeta\ \Delta^2 S \|) .$$

Next we let

$$\varepsilon = \max(|\psi_{,\alpha\beta}|, |w_{,\alpha\beta}|) \tag{2.3}$$

where the maximum is taken over all points in the nondimensional horizontal domain and all values of α, β. Then

$|\Delta^2 \psi| = O(\varepsilon^2)$ and $|\Delta^2 w| = O(\varepsilon^2)$ from (1.2) and (1.3).

We restrict ourselves to the case where $\varepsilon \leq 1$ so that $\varepsilon^2 \leq \varepsilon$. Then from (2.2)

$$\|\zeta \ddddot{\psi}\| = O(\|\ddot{\psi}\| + \|\zeta \Delta^2\psi\|) = O(\varepsilon+\varepsilon^2) = O(\varepsilon) ,$$

$$\|\zeta \ddddot{w}\| = O(\varepsilon) ,$$

$$\|\zeta \overset{\bullet\bullet\bullet}{\psi}\| = O(\|\ddot{\psi}\|) + o(\|\zeta \ddddot{\psi}\|) = O(\varepsilon)$$

$$\|\zeta \overset{\bullet\bullet\bullet}{w}\| = O(\varepsilon) .$$

Next let

$$\bar{\zeta} = \begin{cases} \left(1-\dfrac{4r^2}{\theta^2}\right)^2 & \text{for} \quad 0 \leq r \leq \dfrac{\theta}{2} \\ 0 & \text{for} \quad\quad r \geq \dfrac{\theta}{2} \end{cases}$$

and observe that $\bar{\zeta} = O(\bar{\zeta}\,\zeta)$. Also $1 = O(\zeta)$ in the circle where $\bar{\zeta} \neq 0$.

From (1.3)

$$|\Delta^2\dot{\psi}| = (\varepsilon|\overset{\bullet\bullet\bullet}{w}|)$$

and from (2.2),

$$\|\overset{\bullet\bullet\bullet\bullet\bullet\bullet}{\bar{\zeta}\ \psi}\| = O(\|\overset{\bullet\bullet\bullet}{\psi}\|+\|\bar{\zeta}\ \Delta^2\dot{\psi}\|)$$

where the integration is over the circle where $\bar{\zeta} \neq 0$. Hence

$$\|\overset{\bullet\bullet\bullet\bullet\bullet\bullet}{\bar{\zeta}\ \psi}\| = O(\|\zeta \overset{\bullet\bullet\bullet}{\psi}\|+\|\bar{\zeta}\ \Delta^2\dot{\psi}\|) = O(\varepsilon) .$$

and similarly

$$\|\overset{\bullet\bullet\bullet\bullet\bullet\bullet}{\bar{\zeta}\ w}\| = O(\varepsilon) .$$

From the last line of (2.2) we now have

$$|\overset{\bullet\bullet\bullet}{\psi}|_0 = O(\|\overset{\bullet\bullet\bullet}{\psi}\|+\|\overset{\bullet\bullet\bullet\bullet\bullet\bullet}{\bar{\zeta}\ \psi}\|) = O(\|\zeta \overset{\bullet\bullet\bullet}{\psi}\|+\|\overset{\bullet\bullet\bullet\bullet\bullet\bullet}{\bar{\zeta}\ \psi}\|) = O(\varepsilon) ,$$

where the integration is over the region where $\bar{\zeta} \neq 0$. Similarly $|\dddot{w}|_0 = O(\varepsilon)$.

Continuing in this manner we obtain

$$\left.\begin{aligned} \frac{\partial^k \psi}{\partial x_1^{k_1}\, \partial x_2^{k_2}} &= O(\varepsilon) \\[2ex] \frac{\partial^k w}{\partial x_1^{k_1}\, \partial x_2^{k_2}} &= O(\varepsilon) \end{aligned}\right\} \quad \text{for} \quad k = 2,3,4,5,\ldots\ .$$

where each result is true in some circle with center at $(\hat{x}_1, \hat{x}_2)$.

Since only derivatives of second order or higher of w appear in (1.3), without loss of generality we can take $w_{,\alpha} = 0$ at the origin. Then

$$w_{,\alpha} = \int_0^L w_{,\alpha\beta} \frac{dx_\beta}{ds}\, ds$$

where $x_\beta(s)$ is a path in the domain, s is arc length along the path, and L is the length (nondimensional, of course). Thus $|\dot{w}| = O(\varepsilon)$ where the constant appearing in the order relationship depends on the length of the curve needed to join the point to the origin (i.e. the constant depends on the shape of the region in addition to the distance to the boundary).

Thus

$$(2.4) \qquad \left\{\begin{aligned} \frac{\partial^k \psi}{\partial x_1^{k_1}\, \partial x_2^{k_2}} &= O(\varepsilon) \\[2ex] \frac{\partial^\ell w}{\partial x_1^{\ell_1}\, \partial x_2^{\ell_2}} &= O(\varepsilon) \end{aligned}\right.$$

for $k = 2,3,4,\ldots$ and $\ell = 1,2,3,\ldots$, where the constant appearing in the order relationship depends on the order of the derivatives, the nondimensional distance to the boundary, and the shape of the domain.

The first line of (2.4) is also valid for $k = 0,1$. We obtain this by integrating (1.4) twice along paths in the domain and using (1.5) which holds at the origin. Thus we obtain

$$\frac{\partial^k v_\alpha}{\partial x_1^{k_1}\,\partial x_2^{k_2}} = O(\varepsilon) \tag{2.5}$$

where $k = 0,1,2,3,\ldots$.

The estimates (2.4) and (2.5) are used repeatedly in the following. Since the constants in these order relations generally go to ∞ as the boundary is approached, the results are interior relations with ε as previously defined. We could alternatively define ε so that (2.4) and (2.5) are true in the entire domain with order constants equal "one" for all derivatives up to and including sufficiently high order. Then ε would depend on the derivatives up to sufficiently high order, but would remain independent of h and D. The results obtained hereafter are valid in the entire domain with this alternate value of ε . For this second definition of ε we also require $\varepsilon \leq 1$.

3. Calculation of the Body Forces and Surface Tractions at the Faces.

If we interchange γ and α in (1.4) and add the new equation to (1.4) we obtain

$$(v_{\gamma,\alpha}+v_{\alpha,\gamma})_{,\beta} = (-w_{,\gamma}w_{,\alpha}+2\tau_{\alpha\gamma}-\frac{2\lambda}{3\lambda+2\mu}\,\Delta\psi\,\delta_{\alpha\gamma})_{,\beta}\,.$$

From the above and (1.5) which holds at the origin we have

$$v_{\gamma,\alpha}+v_{\alpha,\gamma} \equiv -w_{,\gamma}w_{,\alpha}+2\tau_{\alpha\gamma}-\frac{2\lambda}{2\lambda+2\mu}\,\Delta\psi\,\delta_{\alpha\gamma}\,. \tag{3.1}$$

From (1.6) and (1.4),

$$\begin{aligned}\mu_{\alpha,\beta} &= \frac{h^2}{D^2}\,(v_{\alpha,\beta}-zw_{,\alpha\beta}) \\ &+ \frac{h^4}{D^4}\Big\{\frac{\lambda}{2(3\lambda+2\mu)}\,z^2\Delta\psi_{,\alpha\beta}+[\frac{3\lambda+4\mu}{6(\lambda+2\mu)}\,z^3-\frac{2(\lambda+\mu)}{\lambda+2\mu}\,z]\,\Delta w_{,\alpha\beta} \\ &+ zw_{,\gamma}(\tau_{\beta\gamma,\alpha}+\tau_{\alpha\gamma,\beta}-\tau_{\alpha\beta,\gamma})+zw_{,\alpha\beta}[\frac{\lambda}{3\lambda+2\mu}\,\Delta\psi-\frac{1}{2}w_{,\gamma}w_{,\gamma}-\frac{\lambda}{2(\lambda+2\mu)}z\,\Delta w] \\ &- \frac{\lambda}{3\lambda+2\mu}\,z(\Delta\psi_{,\alpha}w_{,\beta}-\Delta\psi_{,\gamma}w_{,\gamma}\delta_{\alpha\beta})+zv_{\gamma,\alpha}w_{,\gamma\beta} \\ &+ zw_{,\alpha}[w_{,\gamma}w_{,\gamma\beta}-\frac{\lambda}{2(\lambda+2\mu)}\,z\,\Delta w_{,\beta}]\Big\}\,.\end{aligned} \tag{3.2}$$

Using (3.1) and (3.2) in (1.8) we get

$$\begin{aligned}e_{\alpha\beta} &= \frac{h^2}{D^2}\Big\{\tau_{\alpha\beta}-\frac{\lambda}{3\lambda+2\mu}\,\Delta\psi\,\delta_{\alpha\beta}-zw_{,\alpha\beta}\Big\} \\ &+ \frac{h^4}{D^4}\Big\{\frac{\lambda}{2(3\lambda+2\mu)}\,z^2\Delta\psi_{,\alpha\beta}+[\frac{3\lambda+4\mu}{6(\lambda+2\mu)}\,z^3-\frac{2(\lambda+\mu)}{\lambda+2\mu}\,z]\Delta w_{,\alpha\beta} \\ &+ zw_{,\gamma}(\tau_{\beta\gamma,\alpha}+\tau_{\alpha\gamma,\beta}-\tau_{\alpha\beta,\gamma})+zw_{,\alpha\beta}[\frac{\lambda}{3\lambda+2\mu}\Delta\psi-\frac{1}{2}w_{,\gamma}w_{,\gamma}-\frac{\lambda}{2(\lambda+2\mu)}z\Delta w] \\ &- \frac{\lambda}{3\lambda+2\mu}\,z(\Delta\psi_{,\alpha}w_{,\beta}+\Delta\psi_{,\beta}w_{,\alpha}-\Delta\psi_{,\gamma}w_{,\gamma}\delta_{\alpha\beta})+\frac{1}{2}v_{\gamma,\alpha}v_{\gamma,\alpha}+\frac{z^2}{2}w_{,\gamma\alpha}w_{,\gamma\beta}\Big\} \\ &+ \mathcal{O}(\frac{h^6}{D^6}\,\varepsilon^2)\,.\end{aligned} \tag{3.3}$$

From (1.6) and (1.8)

$$e_{\alpha 3} = \frac{h^3}{D^3}\,\frac{\lambda+\mu}{\lambda+2\mu}\,(z^2-1)\;\Delta w_{,\alpha} + O\Big(\frac{h^5}{D^5}\,\varepsilon^2\Big)\;. \tag{3.4}$$

From (3.1)

$$v_{\gamma,\nu}w_{,\gamma}w_{,\nu} = \frac{1}{2}\,(v_{\gamma,\nu}+v_{\nu,\gamma})\;w_{,\gamma}\;w_{,\nu}$$

$$= w_{,\gamma}w_{,\nu}\Big[-\frac{1}{2}\,w_{,\gamma}w_{,\nu}+\tau_{\gamma\nu}-\frac{\lambda}{3\lambda+2\mu}\,\Delta\psi\;\delta_{\gamma\nu}\Big]\;.$$

From the above, (1.6), and (1.8)

$$\begin{aligned} e_{33} = {} & \frac{h^2}{D^2}\Big(-\frac{\lambda}{3\lambda+2\mu}\,\Delta\psi+\frac{\lambda}{\lambda+2\mu}\,z\;\Delta w\Big) \\ & + \frac{h^4}{D^4}\Big\{-\frac{\lambda}{2(\lambda+2\mu)}\,v_{\gamma,\nu}v_{\gamma,\nu}-\frac{4\mu^2A-\lambda(3\lambda+8\mu)B+\lambda^2C}{(\lambda+2\mu)(3\lambda+2\mu)^2}\,(\Delta\psi)^2 \\ & - \frac{B}{\lambda+2\mu}\,\psi_{,\gamma\nu}\psi_{,\gamma\nu}+\Big[\frac{-\lambda^2}{(\lambda+2\mu)(3\lambda+2\mu)}+\frac{8\mu^2A-12\lambda\mu B+2\lambda^2C}{(\lambda+2\mu)^2(3\lambda+2\mu)}\Big]z\;\Delta\psi\;\Delta w \\ & + \frac{\lambda}{\lambda+2\mu}\,zw_{,\gamma}\Delta\psi_{,\gamma}+\Big[\frac{3\lambda+\mu+2B}{\lambda+2\mu}\,z-\frac{3\lambda+2\mu}{4(\lambda+\mu)}\,z^3\Big]\;w_{,\gamma\nu}\;\tau_{\gamma\nu} \\ & + \Big(3C_3z^2-\frac{\mu}{\lambda+2\mu}\Big)w_{,\gamma\nu}w_{,\gamma\nu}+\frac{\lambda}{2(\lambda+2\mu)}\,z\;\Delta w\;w_{,\gamma}w_{,\gamma} \\ & + \Bigg[\bigg(\frac{\lambda(5\lambda+4\mu)}{8(\lambda+\mu)(\lambda+2\mu)}-\frac{4\mu^2A+\lambda(\lambda-4\mu)B+\lambda^2C}{(\lambda+2\mu)^3}\,z^2-\frac{\lambda\mu}{(\lambda+2\mu)^2}\Bigg]\,(\Delta w)^2\Big\} \\ & + O\Big(\frac{h^6}{D^6}\,\varepsilon^2\Big)\;. \end{aligned} \tag{3.5}$$

From (3.3) and (1.3),

$$\begin{aligned} e_{\gamma\gamma} = {} & \frac{h^2}{D^2}\Big(\frac{\lambda+2\mu}{3\lambda+2\mu}\,\Delta\psi-z\;\Delta w\Big) \\ & + \frac{h^4}{D^4}\Big\{\frac{1}{2}\,v_{\gamma,\nu}v_{\gamma,\nu}+\Big[\frac{3\lambda+4\mu}{4(\lambda+\mu)}\,z^3-3z\Big]w_{,\gamma\nu}\tau_{\gamma\nu} \\ & + z\;\Delta w\;\Big(\frac{\lambda}{3\lambda+2\mu}\,\Delta\psi-\frac{1}{2}\,w_{,\gamma}w_{,\gamma}\Big)-zw_{,\gamma}\Delta\psi_{,\gamma} \\ & + \frac{5\lambda+4\mu}{8(\lambda+\mu)}\,z^2w_{,\gamma\nu}w_{,\gamma\nu}-\frac{\lambda(5\lambda+6\mu)}{8(\lambda+\mu)(\lambda+2\mu)}\,z^2(\Delta w)^2\Big\}+O\Big(\frac{h^6}{D^6}\,\varepsilon^2\Big)\;. \end{aligned} \tag{3.6}$$

Using (3.3,4,5,6) in (1.10), we obtain

$$
\begin{aligned}
t_{\alpha\beta} &= \frac{h^2}{D^2}\{2\mu\tau_{\alpha\beta} - 2\mu z\left(w_{,\alpha\beta} + \frac{\lambda}{\lambda+2\mu}\delta_{\alpha\beta}\Delta w\right)\} \\
&\quad + \frac{h^4}{D^4}\left\{\frac{\lambda\mu}{3\lambda+2\mu} z^2 \Delta\psi_{,\alpha\beta} + \left[\frac{\mu(3\lambda+4\mu)}{3(\lambda+2\mu)} z^3 - \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} z\right]\Delta w_{,\alpha\beta}\right\} \\
&\quad + \mathcal{O}\left(\mu\frac{h^4}{D^4}\varepsilon^2\right) \qquad (3.7) \\
t_{\alpha 3} &= \frac{h^3}{D^3}\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}(z^2-1)\Delta w_{,\alpha} + \mathcal{O}\left(\mu\frac{h^5}{D^5}\varepsilon^2\right) \\
t_{33} &= \frac{h^4}{D^4}\mu(1-z^2)\left[z w_{,\gamma\nu}\tau_{\gamma\nu} - w_{,\gamma\nu}w_{,\gamma\nu} - \frac{\lambda}{\lambda+2\mu}(\Delta w)^2\right] + \mathcal{O}\left(\mu\frac{h^6}{D^6}\varepsilon^2\right)
\end{aligned}
$$

From (1.10), (1.6), and (3.7), we have

$$
\begin{aligned}
q_{\alpha\beta} &= t_{\alpha\beta} + \mathcal{O}\left(\mu\frac{h^4}{D^4}\varepsilon^2\right) \\
q_{\alpha 3} &= t_{\alpha 3} + \mathcal{O}\left(\mu\frac{h^5}{D^5}\varepsilon^2\right) \qquad (3.8) \\
q_{3\alpha} &= t_{3\alpha} + \frac{h^3}{D^3}\left[2\mu w_{,\gamma}\tau_{\gamma\alpha} - 2\mu z\left(w_{,\gamma}w_{,\gamma\alpha} + \frac{\lambda}{\lambda+2\mu}w_{,\alpha}\Delta w\right)\right] + \mathcal{O}\left(\mu\frac{h^5}{D^5}\varepsilon^2\right) \\
q_{33} &= t_{33} + \frac{h^4}{D^4}\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}(z^2-1)w_{,\gamma}\Delta w_{,\gamma} + \mathcal{O}\left(\mu\frac{h^6}{D^6}\varepsilon^2\right).
\end{aligned}
$$

From (1.11), (3.7), (3.8), (1.3) we have

$$
\begin{aligned}
F_\alpha &= \mathcal{O}\left(\mu\frac{h^4}{D^5}\varepsilon^2\right) \qquad (3.9) \\
F_3 &= \mathcal{O}\left(\mu\frac{h^5}{D^6}\varepsilon^2\right) .
\end{aligned}
$$

Also from (1.11), (3.7), (3.8)

$$
\begin{aligned}
T_\alpha &= \mathcal{O}\left(\mu\frac{h^5}{D^5}\varepsilon^2\right) \qquad (3.10) \\
T_3 &= \mathcal{O}\left(\mu\frac{h^6}{D^6}\varepsilon^2\right) .
\end{aligned}
$$

on the faces.

This completes the verification of (0.1), the objective of the report.

As mentioned previously our estimates are not valid at the edge if our first definition of ε is used. However, if our second definition of ε is used, all results are valid at the edge and we can compute the stress vector at the edge. Letting (S_1, S_2, S_3) be the stress vector at the edge, we have

$$S_i = q_{i\gamma} n_\gamma$$

where $(n_1, n_2, 0)$ is the unit outer normal to the undeformed edge. Then from (3.8) and (3.7)

$$S_\alpha = t_{\alpha\beta} n_\beta + O\left(\mu \frac{h^4}{D^4} \varepsilon^2\right) = \frac{h^2}{D^2} \left\{ 2\mu\tau_{\alpha\beta} n_\beta - 2\mu z \left(\frac{dw_{,\alpha}}{dn} + \frac{\lambda}{\lambda+2\mu} n_\alpha \Delta w \right) \right.$$

$$+ O\left(\mu \frac{h^4}{D^4} \varepsilon\right) \quad \text{where} \quad \frac{d}{dn} \quad \text{is the normal derivative}$$

in the outer direction

$$S_3 = O\left(\mu \frac{h^3}{D^3} \varepsilon\right) .$$

A common set of boundary conditions for the von Karman equations is to prescribe the edge thrust $\tau_{\alpha\beta} n_\beta$ and zero moment around the edge, namely, $\frac{dw_{,\alpha}}{dn} + \frac{\lambda}{\lambda+2\mu} n_\alpha \Delta w = 0$. From the formula for S_α , this amounts to prescribing the horizontal components of the stress vector at the edge to within terms of order $\mu \frac{h^4}{D^4} \varepsilon$. Of course the vertical component of the stress vector is zero at the edge to within order $\mu \frac{h^3}{D^3} \varepsilon$ from the formula for S_3.

Finally, I wish to thank Professor J. Stoker for making valuable suggestions in the writing of this report, and I wish to acknowledge the work of Professor F. John which opened the door to work of this type in elasticity.

References

[1] John, F., *Estimates for the derivatives of the stresses in a thin shell and interior shell equations*, Comm. Pure Appl. Math., Vol. 18, 1965, pp. 235-267.

[2] Sensenig, C., *Plate equations and formulas for stresses, strains, and displacements*, New York University, Courant Insttitute of Mathematical Sciences, Res. Rep. IMM 395 (1973).

[3] Thomas, T. Y., *Systems of total differential equations defined over simply connected domains*, Annals of Math., Vol. 35, pp. 730-734 (1934).

[4] Truesdell, C. and Noll, W., *The nonlinear field theories of mechanics*, Handbuch der Physik, Vol. III/3, Springer-Verlag, 1965.

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
IMM 403

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE *(and Subtitle)*
Construction of a Three Dimensional Displacement Field from a Solution to the von Karman Equations

5. TYPE OF REPORT & PERIOD COVERED
Technical

6. PERFORMING ORG. REPORT NUMBER
IMM 403

7. AUTHOR*(s)*
Chester B. Sensenig

8. CONTRACT OR GRANT NUMBER*(s)*
N00014-67-A-0467-0030

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Courant Inst. of Mathematical Sciences
New York University
251 Mercer St., New York, N.Y. 10012

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
NR 041-446

11. CONTROLLING OFFICE NAME AND ADDRESS
Mathematics Branch
Office of Naval Research
Washington D.C. 20360

12. REPORT DATE
June 1974

13. NUMBER OF PAGES
20

14. MONITORING AGENCY NAME & ADDRESS*(if different from Controlling Office)*

15. SECURITY CLASS. *(of this report)*
not classified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
none

16. DISTRIBUTION STATEMENT *(of this Report)*

Distribution of this document is unlimited.

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

none

18. SUPPLEMENTARY NOTES

none

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

none

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

In this report we define a three dimensional displacement field in terms of a solution to the von Karman equations and determine how well the three dimensional displacement field satisfies the exact theory of elasticity for a plate with zero body force and zero stress vector at the faces. Letting F_1, F_2, F_3 be the rectangular components of the body force vector (force

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE per unit undeformed volume) and letting T_1, T_2, T_3 be the rectangular components of the stress vector at the faces (force per unit undeformed area), we obtain

$$F_1, F_2 = O\left(\frac{\mu h^4}{D^5} \varepsilon^2\right), \quad F_3 = O\left(\frac{\mu h^5}{D^6} \varepsilon^2\right),$$

$$T_1, T_2 = O\left(\frac{\mu h^5}{D^5} \varepsilon^2\right), \quad T_3 = O\left(\frac{\mu h^6}{D^6} \varepsilon^2\right).$$

For the above, the plate is horizontal; F_1, F_2, T_1, T_2 are horizontal components; and F_3, T_3 are vertical components. The undeformed plate thickness is 2h, the undeformed plate diameter is D, μ is a Lame constant, and ε is the maximum absolute value of the second derivatives of the solution to the nondimensional von Karman equations. The parameters h, D, and ε are independent.

This shows how rapidly the body force and stress vector at the faces approach zero values as the solution to the von Karman equations and the thickness to diameter ratio approach zero values.

This book may be kept JUL 19 1974
FOURTEEN DAYS
A fine will be charged for each day the book is kept overtime.
GAYLORD 142 PRINTED IN U.S.A.